A brief over-view of Fixed Point Theory

Theory and Applications of Fixed Points

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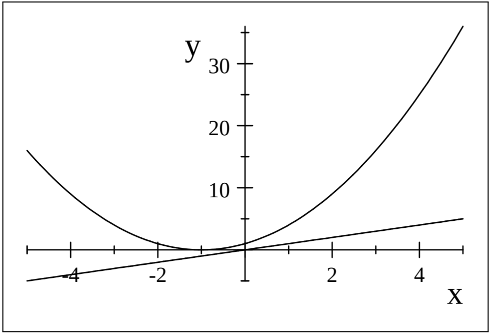
- For instance, $x = -1$ in $y = 3x + 2$
- A fixed point may not exist, may exist but may not be unique
- Used to find roots of an equation as follows: write out $f(x)$ in the form $g(x) = x$ or by finding fixed points of $g(x) = x + f(x)$

Graphical Illustrations

$$f(x) = x^2 + 2x + 1$$



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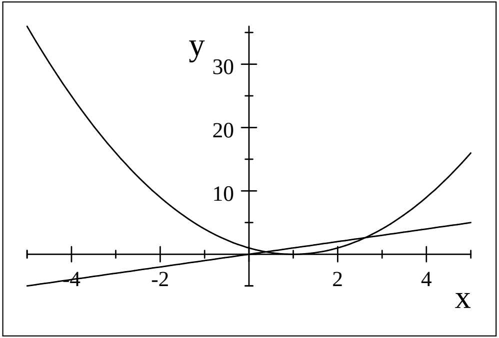

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- f is a **contraction** if $\alpha \in (0, 1)$

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- $f(x) = x$ is nonexpansive but not contractive since $d(f(x), f(y)) = d(x, y)$
- For $X = (1, \infty)$, $f(x) = x + 1/x$ is contractive but not a contraction since $d(f(x), f(y)) =$
  $|(x - y) + (1/x - 1/y)|$
  $= |x - y| |1 - 1/xy| < |x - y|$

Preliminary theorems

Theorem

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Proof.

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$d(x, y) < \delta \implies d(f(x), f(y)) \leq \alpha d(x, y) < \epsilon$ □

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For $X = \mathbb{R}$ *and usual metric* d, $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is a contraction* $\iff |g'(x)| \leq \alpha < 1$ *for continuous* g.

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Proof.

$(\impliedby)$ $\frac{g(x)-g(y)}{x-y} = g'(t)$ for $t \in (x - y - \delta, x - y + \delta)$ from MVT $\implies |g(x) - g(y)| \le \alpha |x - y|$
$(\implies)$ $|g(x + h) - g(x)| \le \alpha |h|$ □

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Fact

Invalid if completeness or contraction condition is taken

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$$g(x) = x + \frac{f(x)}{f'(x)} \implies g'(x) = 1 = f(x)\, f''(x) \,/\, [f'(x)]^2 \implies \lim_{x \longrightarrow \hat{x}} g'(x) = 0 \implies |g'(x)| < \epsilon$$

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- $x_{n+1} = g(x_n) = x_n + \frac{f(x_n)}{f'(x_n)}$

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Application of Banach's Theorem

Finding roots of equations

- What if we need to find $x = \sqrt[r]{c}$?
- Take $f(x) = x^r - c = 0$.
- Apply Newton's formula to get $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)} = g(x_n) = \frac{1}{r}\left(x_n + \frac{c}{x_n}\right)$.

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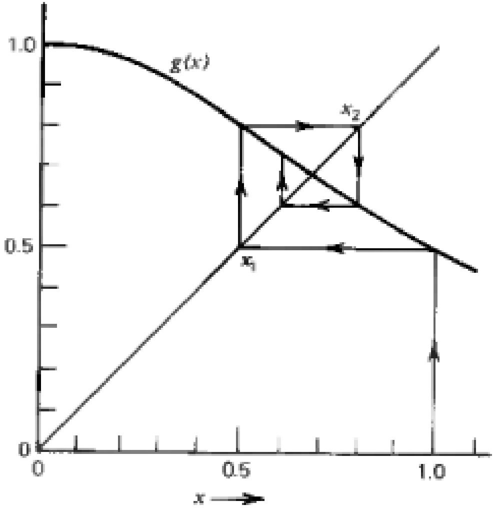
- This map is contraction with $\alpha = |1/r\,(1 - c/xy)|$

Point



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Application of Banach's Theorem

Finding solution for system of equations

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- What about $A\mathbf{x} = \mathbf{b}$? Convert to $D\mathbf{x} = (D - A)\mathbf{x} + b$ and use previous condition

Theorem (Picard's Theorem)

If $|f(t,x) - f(t,y)| \leq \alpha |x - y|$, *then* $x'(t) = f(t, x(t))$ *has a solution in some interval with initial value* $x(t_0) = x_0$

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Turn IVP into $T(x(t)) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$ □

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If f_x *exists, then* $|f(t,x) - f(t,y)| \leq \alpha |x-y|$

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A continuous image of a compact set $M \subseteq \mathbb{R}^n$ is compact

From Metric Spaces to Topological Spaces

Theorem

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Proof.

If $x_n \longrightarrow x \in M$ and $x_{n_k} \longrightarrow x$, then $f(x_{n_k}) \longrightarrow f(x)$ □

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U *and* $f\left(U\right)$ *are homeomorphic*

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Corollary

U and $f(U)$ are homeomorphic

Corollary

A continuous injective map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ does not exist for $n \neq m$

Definition

A continuous map f from a topological space T to a subspace S is called a **retraction** if $f(x) = x \ \forall x \in S$

Example

The retract of $[0, 1]$ is not $(0, 1)$

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Every continuous mapping $f : \overline{B_n(0, 1)} \longrightarrow \overline{B_n(0, 1)}$ *has a fixed point.*

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For $n = 1$, $f(-1) \geq (-1)$ and $f(1) \leq 1$ so that $g(x) = x - f(x)$ has a solution $g(c) = 0$ for $c \in [-1, 1]$. For arbitrary n, take $g(x) = \alpha(x)x + (1 - \alpha(x))f(x)$ with $g(x) \in S_n(0,1)$, which is well defined if f does not have a fixed point but g cannot exist □

From Metric Spaces to Topological Spaces

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Proof.

For $h : Y \longrightarrow Y$, if $f : X \longrightarrow X$, then $f = g^{-1} \circ h \circ g \implies h$ has a fixed point □

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for $n = 1$, $f : [-1,1] \longrightarrow \{-1,1\}$ is not differentiable
If $r : \overline{B_n(0,1)} \longrightarrow S_n(0,1)$ is a differentiable retraction, define $r_1 : \overline{B_n(0,1)} \longrightarrow \overline{B_n(0,1)}$ such that $r_1(x) = -r(x)$. Then, $r_1(x)$ has a fixed point by Brouwer's theorem but $r_1(x) = x = -r(x)$ is then not a retraction □

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Example (Failure)

$l^2 \ni x = (x_1, x_2, ...) \longmapsto (1 + \|x\|, x_1, x_2, ...)$

Going high up the dimension ladder

Theorem (Schauder's fixed point theorem)

A continuous self-map on a compact, convex set in a Banach space has a fixed point